# Audio Signal Processing : IV. Stochastic signal processing

Emmanuel Bacry

bacry@ceremade.dauphine.fr
http://www.cmap.polytechnique.fr/~bacry

**Framework** :

$X[n]$ is a stochastic disrete-time signal

$X[n]$ is real-valued

**Stochastic signal : What for ?**

**Discrete-time stationnary processes : What do we need them for ?**

**What do we mean by stationarity ?**

**Strict-sense stationnarity** :

$X[n]$ : discrete-time stochastic procecss

$$\forall p \in \mathbb{N},\ \forall (n_1, \ldots, n_p) \in \mathbb{Z}^p,\ \forall k \in \mathbb{Z},$$

$$\{X_{n_1}, \ldots, X_{n_p}\} \overset{law}{=} \{X_{n_1+k}, \ldots, X_{n_p+k}\}$$

**Second-order (wide-sense) stationnarity** :

- $X[n]$ : discrete-time stochastic process
- $\mathrm{Var}(X[n]) < +\infty$
- $\forall n, \ \mathrm{E}(X[n]) = \mu$
- $\forall n, \ \forall k, \quad \mathrm{Cov}(X[n], X[n+k]) = R_X[k]$

**Theorem**

A wide-sense stationnary Gaussian process is strict-sense stationnary

**Problem** : We want to estimate a deterministic quantity $y$ that is a function of $\{X[n]\}_n$

The **estimator** is a r. v. $Y_N$ that is a function of $\{X[n]\}_{0 \leq N < N}$

The "quality" of the estimator is often quantified by the MSE

$$MSE(N) = E((y - Y_N)^2)$$

**Example**

- $y = \mathrm{E}(X[n])$
- $Y_N = \frac{1}{N} \sum_{k=0}^{N-1} X[k]$

**Problem** : $y$ is estimated using the estimator $Y_N$

**Two important quantities**

- **Bias** : $Bias_N = E(y - Y_N)$
  $\longrightarrow$ (asymptotically) unbiased estimator ?
- **Variance** : $Var_N = E(Y_N^2) - E(Y_N)^2$
  $\longrightarrow$ consistent estimator ?

**Theorem**

$$MSE(N) = Bias_N^2 + Var_N$$

**An estimation example** (Mean estimation):

- $y = E(X[n])$
- $Y_N = \frac{1}{N}\sum_{n=0}^{N-1} X[k]$ (ergodicity)

Let's study

- Bias ?
- Consistency ?

**An estimation example** (Mean estimation):

- $y = E(X[n]) = E(X)$
- $Y_N = \frac{1}{N}\sum_{n=0}^{N-1} X[k]$ (ergodicity)

**Bias** :

$$\begin{aligned}
E(Y_N - y) &= E\left(\frac{1}{N}\sum_{n=0}^{N-1} X[k] - E(X[n])\right) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} E(X[k]) - E(X[n]) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} E(X) - E(X) \\
&= 0
\end{aligned}$$

This is un unbiased estimator

**An estimation example** (Mean estimation):

- $y = E(X[n]) = E(X)$
- $Y_N = \frac{1}{N}\sum_{n=0}^{N-1} X[k]$ (ergodicity)

**Consistency (1/2)** :

$$
\begin{aligned}
E(Y_N^2) - E(Y_N)^2 &= E\left(\left(\frac{1}{N}\sum_{n=0}^{N-1} X[k]\right)^2\right) - E(X)^2 \\
&= E\left(\frac{1}{N^2}\sum_{n,k=0}^{N-1} X[k]X[n]\right) - E(X)^2 \\
&= \frac{1}{N^2}\sum_{n,k=0}^{N-1} E(X[k]X[n]) - E(X)^2 \\
&= \frac{1}{N^2}\sum_{n,k=0}^{N-1} R_X[n-k]
\end{aligned}
$$

**An estimation example** (Mean estimation):

- $y = E(X[n]) = E(X)$
- $Y_N = \frac{1}{N}\sum_{n=0}^{N-1} X[k]$ (ergodicity)

**Consistency (2/2)** :

$$
\begin{aligned}
E(Y_N^2) - E(Y_N)^2 &= \frac{1}{N^2}\sum_{n,k=0}^{N-1} R_X[n-k] \\
&= \frac{1}{N}R_X[0] + \frac{2}{N^2}\sum_{l=1}^{N-1}(N-|l|)R_X[l] \\
&= \frac{1}{N}R_X[0] + \frac{2}{N}\sum_{l=1}^{N-1}(1-|l|/N)R_X[l]
\end{aligned}
$$

Thus if $R_X[n] \in l^1$ the operator is consistent

**Another estimation example** (Covariance estimation):

- $y = R_X[k]$ (we suppose $E(X) = 0$)
- (using ergodicity) $Y_N = \frac{1}{N-k} \sum_{n=0}^{N-1-k} X[n]X[n+k]$

Let's study

- Bias ?
- Consistency ?

**Another estimation example** (Covariance estimation):

- $y = R_X[k]$ (we suppose $E(X) = 0$)
- (using ergodicity) $Y_N = \frac{1}{N-k}\sum_{n=0}^{N-1-k} X[n]X[n+k]$

**Bias** :

$$
\begin{aligned}
E(Y_N - y) &= E\left(\frac{1}{N-k}\sum_{n=0}^{N-1-k} X[n]X[n+k]\right) - R_X[k] \\
&= \frac{1}{N-k}\sum_{n=0}^{N-1-k} E(X[n]X[n+k]) - R_X[k] \\
&= \frac{1}{N-k}\sum_{n=0}^{N-1-k} R_X[k] - R_X[k] \\
&= 0
\end{aligned}
$$

This is an unbiased estimator

**Another estimation example** (Covariance estimation):

- $y = R_X[k]$ (we suppose $E(X) = 0$)
- (using ergodicity) $Y_N = \frac{1}{N-k} \sum_{n=0}^{N-1-k} X[n]X[n+k]$

**Consistency ?**

**Another estimation example** (A "better" covariance estimator):

- $y = R_X[k]$
- $Y_N = \frac{1}{N} \sum_{n=0}^{N-1-k} X[n]X[n+k]$

- Asymptotically unbiased estimator
- Consistent estimator (if $R_X[k]$ is decreasing quickly enough)

**Definition** : it is a **positive** bilinear operator

- $A = \sum_{i=1}^{N} a_i X[i]$
- $B = \sum_{i=1}^{N} b_i X[i]$

$$\mathrm{Cov}(A, B) = \sum_{i,j} a_i R_X[i-j] b_j = a.(R \star b) = a.L(B)$$

$\Longrightarrow$ The associated linear form the convolution operator

$$L(B) = R \star b$$

**Problem :**

**How to define the Fourier transform of a stochastic stationnary process ?**

**Definition of Power spectrum**

$$\hat{R}_X(e^{i\omega}) = \sum_n R_X[n] e^{in\omega}$$

- Why is it real ?
- Why is it positive ?
- What is the inverse Fourier transform ?

**An interpretation of the Power spectrum**

$\hat{R}_X(e^{i\omega})$ represents the average energy contained (in average) by a realization at frequency $\omega$

$$R_X[k] = \frac{1}{2\pi} \int_0^{2\pi} \hat{R}_X(e^{i\omega}) e^{ik\omega} d\omega$$

**An important example : The white noise**

- $X[n]$ second-order process
- $R_X[k] = \sigma^2 \delta[k]$
- $\hat{R}_X(e^{i\omega}) = 1$

**What Estimator for the power spectrum ?**

A "natural" estimator could be the **Periodogram**

$$\tilde{\hat{R}}_X(e^{i\omega}) = |\sum_{n=0}^{N-1} X[n]e^{-in\omega}|^2$$

?

**A consistent estimator for the power spectrum : The averaged periodogram**

$$\tilde{\hat{R}}_X(e^{i\omega}) = \frac{K}{N-1} \sum_{n=0}^{(N-1)/K-1} \left| \sum_{k=nK}^{(n+1)K-1} X[k] e^{-ik\omega} \right|^2$$

## Towards the Convolution Theorem

Let $\{X[n]\}_n$ a discrete-time second order stationary process with $\mathrm{E}(X[n]) = 0$ then

- $\forall h \in l^1$, we define $h \star X[n] = \sum_k h[n-k]X[k]$
- $\mathrm{E(h \star X[n])} = \sum_k h[n-k]\mathrm{E}(X[k]) = 0$
- $\forall h \in l^1$, $\forall g \in l^1$, one gets

$$\forall n, n', \quad \mathrm{Cov}(h \star X[n]), g \star X[n']) = R_X \star g \star \tilde{h}[n'-n]$$

- $\forall h \in l^1$, one gets

$$\forall n, k, \quad \mathrm{Cov}(h \star X[n]), h \star X[n+k]) = R_X \star h \star \tilde{h}[k]$$

**The Convolution Theorem**

Let $\{X[n]\}_n$ a discrete-time second order stationary process with $\mathrm{E}(X[n]) = 0$ then $\forall h \in l^1$, one gets

$$\hat{R}_{h\star X}(e^{i\omega}) = |\hat{h}(e^{i\omega})|^2 \hat{R}_X(e^{i\omega})$$

$$\hat{R}_X(e^{i\omega}) = \sum_n R_X[n]e^{in\omega}$$

**Energy formula**

- Energy in the time-domain $E(X[n]^2) = R_X[0]$
- Energy in the frequency-domain

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{R}_X(e^{i\omega})d\omega$$

Energy conservation (i.e., inverse Fourier transorm or $n = 0$)

$$R_X[0] = \frac{1}{2\pi}\int_0^{2\pi} \hat{R}_X(e^{i\omega})d\omega$$

**Property**

- Let $h_{\omega_0,\Delta\omega}[n]$ be the band pass filter such that

$$\hat{h}_{\omega_0,\Delta\omega}|_{]-\pi,\pi[} = \sqrt{\frac{\pi}{\Delta\omega}}\left(1_{]-\omega_0-\Delta\omega,-\omega_0+\Delta\omega[}(\omega) + 1_{]\omega_0-\Delta\omega,\omega_0+\Delta\omega[}(\omega)\right)$$

- Let $X_{\omega_0,\Delta\omega}[n] = h_{\omega_0,\Delta\omega} \star X[n]$

What is the energy of the limit process

$$\lim_{\Delta\omega\to 0} X_{\omega_0,\Delta\omega}[n] \quad ?$$

$$\hat{h}_{\omega_0,\Delta\omega}|_{]-\pi,\pi[} = \sqrt{\frac{\pi}{\Delta\omega}}\left(1_{]-\omega_0-\Delta\omega,-\omega_0+\Delta\omega[}(\omega) + 1_{]\omega_0-\Delta\omega,\omega_0+\Delta\omega[}(\omega)\right)$$

Energy of $X_{\omega_0,\Delta\omega}[n]$ writes

$$\begin{aligned}
R_{X_{\omega_0,\Delta\omega}}[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{R}_{X_{\omega_0,\Delta\omega}}(e^{i\omega})d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{h}_{\omega_0,\Delta\omega}(e^{i\omega})|^2\hat{R}_X(e^{i\omega})d\omega \\
&= \frac{1}{2\Delta\omega}\left(\int_{-\omega_0-\Delta\omega}^{-\omega_0+\Delta\omega}\hat{R}_X(e^{i\omega})d\omega + \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}\hat{R}_X(e^{i\omega})d\omega\right) \\
&\longrightarrow \hat{R}_X(e^{-i\omega_0}) + \hat{R}_X(e^{i\omega_0})
\end{aligned}$$